Strictly Complementary Solutions in Linear Programming

by

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Agenda

- Background
- Examples of Information Value
- Some Conclusions
- New Results and Further Research

References:

H.J. Greenberg. The Use of the Optimal Partition in a Linear Programming Solution for Postoptimal Analysis, *OR Letters* 15:4 (1994) 179–185.

H.J. Greenberg. Rim Sensitivity Analysis from a Strictly Complementary Solution, *SIOPT* (to appear).

Background

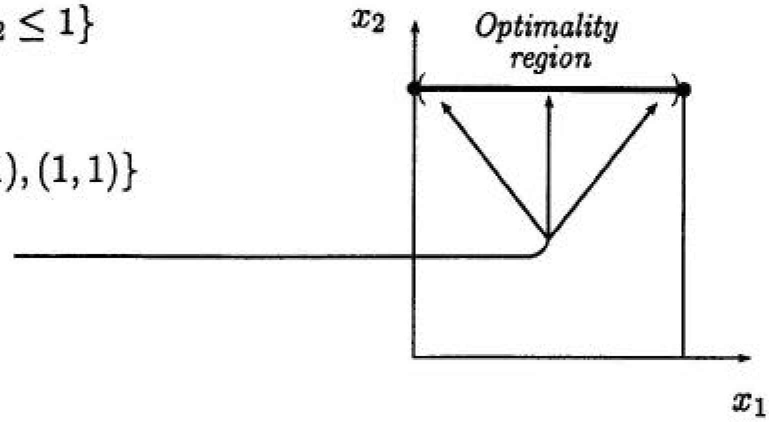
What is an interior solution?

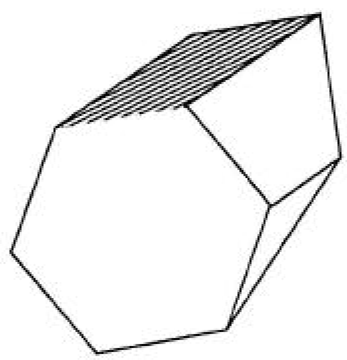
$\text{argmax}\{x_2 : 0 \le x_1, x_2 \le 1\}$

$= \{(\zeta, 1) : 0 \le \zeta \le 1\}$

Extreme points: $\{(0,1), (1,1)\}$

Relative interior:
$\{(\zeta, 1) : 0 < \zeta < 1\}$





If you don't know where you're going, you'll probably end up somewhere else.
— Casey Stengel

Strict Complementarity

Primal	Dual
$\min cx:\ x \geq 0, Ax \geq b$	$\max \pi b:\ \pi \geq 0, \pi A \leq c$
$s \equiv Ax - b\ (\geq 0)$	$d \equiv c - \pi A\ (\geq 0)$

For x feasible in primal and π feasible in dual,

$$\textit{Duality gap} \ \equiv cx - \pi b = dx + \pi s \geq 0.$$

(x, π) optimal $\Leftrightarrow$ Duality gap $= 0$ $\Leftrightarrow$ *complementary:*

$$x_j > 0 \Rightarrow d_j = 0; \quad d_j > 0 \Rightarrow x_j = 0;$$
$$s_i > 0 \Rightarrow \pi_i = 0; \quad \pi_i > 0 \Rightarrow s_i = 0.$$

Could have $x_j = d_j = 0$ and/or $s_i = \pi_i = 0$ for any *complementary pair.*

Strictly complementary:

$$x_j = 0 \Rightarrow d_j > 0; \quad d_j = 0 \Rightarrow x_j > 0;$$
$$s_i = 0 \Rightarrow \pi_i > 0; \quad \pi_i = 0 \Rightarrow s_i > 0.$$

Logic is the art of going wrong with confidence.

— Joseph Wood Krutch

Example Revisited

$$\begin{aligned} &\text{argmax}\{x_2 : x_1, x_2 \geq 0 \\ &\quad x_1,\ x_2 \leq 1\} \\ &= \{(\zeta, 1) : 0 \leq \zeta \leq 1\} \end{aligned}$$

$$\begin{aligned} &\text{argmin}\{\pi_1 + \pi_2 : \pi_1, \pi_2 \geq 0 \\ &\quad \pi_2 \geq 1\} \\ &= \{(0, 1)\} \end{aligned}$$

(Dual solution is unique)

$$s = b - Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - I \begin{pmatrix} \zeta \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \zeta \\ 0 \end{pmatrix};$$

$$d = c - \pi A = (0,\ 1) - (0,\ 1)I = (0,\ 0).$$

Complementary Pairs

$x \perp d$				$s \perp \pi$			
x_1	d_1	x_2	d_2	s_1	π_1	s_2	π_2
ζ	0	1	0	$1 - \zeta$	0	0	1

Strictly complementary $\Leftrightarrow 0 < \zeta < 1$.

Economic Theory: A systematic application and critical evaluation of the basic analytic concepts of economic theory, with an emphasis on money and why it's good.

— Woody Allen

Support Sets

Support set = coordinates for which value is positive:

$$\begin{aligned} \sigma(x) &= \{j: x_j > 0\}, & \sigma(s) &= \{i: s_i > 0\} \\ \sigma(d) &= \{j: d_j > 0\}, & \sigma(\pi) &= \{i: \pi_i > 0\}. \end{aligned}$$

Complementary: $\sigma(x) \cap \sigma(d) = \emptyset$; $\sigma(s) \cap \sigma(\pi) = \emptyset$.

$\Leftrightarrow$ Exclusive

Strictly complementary: $\sigma(x) \cup \sigma(d) = \{1, \ldots, n\}$; $\sigma(s) \cup \sigma(\pi) = \{1, \ldots, m\}$.

$\Leftrightarrow$ + Exhaustive

A strictly complementary solution induces a *partition.*

Key Fact

☺ Every LP that has an optimal solution has a strictly complementary solution, and the partition induced by every strictly complementary solution is the same [Goldman and Tucker, 1956].

We thus refer to ***the*** optimal partition of the (primal-dual) LP, which is obtained by *any* strictly complementary solution.

Same example: For *any* strictly complementary solution,

$$\begin{aligned} \sigma(x) &= \{1, 2\} & \sigma(s) &= \{1\} \\ \sigma(d) &= \emptyset & \sigma(\pi) &= \{2\} \end{aligned}$$

If I had enough time, I could write less.

— B. Pascal

Facts About the (Unique) Optimal Partition

- Typical interior point methods (viz., central path following) converge to a strictly complementary solution [Adler and Monteiro, 1989, -92; Güler, Roos, Terlaky and Vial, 1992; Jansen, Roos and Terlaky, 1992].

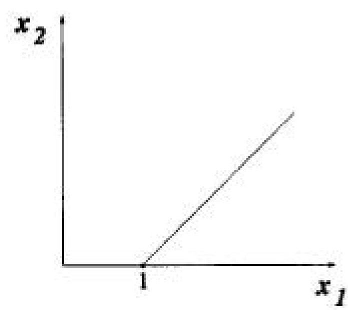
- A basic optimal solution is strictly complementary with its associated (optimal) dual prices if, and only if, it is the only optimal solution (for both primal and dual) [Greenberg, 1986].

Caution: There can be only one basic optimum, but still be alternative optima.
Example:

Primal	Dual
$\min 0x : x \geq 0, x_1 - x_2 \geq 1$	$\max \pi : \pi \geq 0, \pi \leq 0.$

Strictly complementary solution: $x^0 = (3,1), d^0 = (0,0), s^0 = 1, \pi^0 = 0.$

Basic optimal solution: $x^1 = (1,0), d^1 = (0,0), s^1 = 0, \pi^1 = 0.$



If optimality region (both primal and dual) is bounded:

unique optimum $\Leftrightarrow$ unique optimal basis $\Leftrightarrow$ strictly complementary.

The search for truth is more precious than its possession.

— Albert Einstein

An Example of Rim Ranges

3×3 transportation problem:

$$\min \sum_{ij} c_{ij} x_{ij} : x \geq 0, \sum_j x_{ij} \leq a_i, \sum_i x_{ij} \geq b_j.$$

Current values: $c_{ij} = 1 \, \forall i, j$; $a = (2, 6, 5)$; $b = (3, 3, 3)$.

	Primal Solution									Dual Solution					
System	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{31}	x_{32}	x_{33}	π_1	π_2	π_3	π_4	π_5	π_6
CPLEX	0	2	0	2	1	3	1	0	0	0	0	0	1	1	1
LINDO	2	0	0	0	0	2	1	3	1	0	0	0	1	1	1
PC-PROG	0	0	0	0	3	1	3	0	2	0	0	0	1	1	1
XMP	0	0	2	3	3	0	0	0	1	0	0	0	1	1	1

	RHS Ranges					
System	a_1	a_2	a_3	b_1	b_2	b_3
CPLEX	$[0,3]$	$[4,7]$	$[1,\infty)$	$[2,7]$	$[2,5]$	$[2,5]$
LINDO	$[1,3]$	$[2,\infty)$	$[4,7]$	$[2,4]$	$[1,4]$	$[1,7]$
PC-PROG	$[0,\infty)$	$[4,\infty)$	$[3,6]$	$[2,5]$	$[0,5]$	$[2,5]$
XMP	$[0,3]$	$[6,7]$	$[1,\infty)$	$[2,3]$	$[2,3]$	$[2,7]$

	Cost Ranges								
System	c_{11}	c_{12}	c_{13}	c_{21}	c_{22}	c_{23}	c_{31}	c_{32}	c_{33}
CPLEX	$[1,\infty)$	$(-\infty,1]$	$[1,\infty)$	$[1,1]$	$[1,1]$	$[0,1]$	$[1,1]$	$[1,\infty)$	$[1,\infty)$
LINDO	$(-\infty,1]$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,1]$	$[1,1]$	$[0,1]$	$[1,1]$
PC-PROG	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[1,\infty)$	$[0,1]$	$[1,1]$	$[0,1]$	$[1,\infty)$	$[1,1]$
XMP	$[1,\infty)$	$[1,\infty)$	$(-\infty,1]$	$[0,1]$	$[0,1]$	$[1,1]$	$[1,\infty)$	$[1,\infty)$	$[1,1]$

Reference:

B. Jansen, C. Roos and T. Terlaky. *An Interior Point Approach to Postoptimal and Parametric Analysis in Linear Programming*, Technical Report, Faculty of Technical Mathematics and Informatics/Computer Science, Delft University of Technology, Delft, The Netherlands, 1992.

Ranges from the Interior Method

RHS Ranges

	a_1	a_2	a_3	b_1	b_2	b_3
Min	0	2	1	0	0	0
Current	2	6	5	3	3	3
Max	∞	∞	∞	7	7	7

Cost Ranges

	c_{11}	c_{12}	c_{13}	c_{21}	c_{22}	c_{23}	c_{31}	c_{32}	c_{33}
Min	$-\infty$	$-\infty$	$-\infty$	0	0	0	0	0	0
Current	1	1	1	1	1	1	1	1	1
Max	∞	∞	∞	∞	∞	∞	∞	∞	∞

- These ranges are **unique**.
- = *Break points*: objective value changes form.
- = where the optimal partition must change.

Even if you're on the right track, you'll get run over if you just sit there.

— Will Rogers

Meaning of the Optimal Partition Support Sets

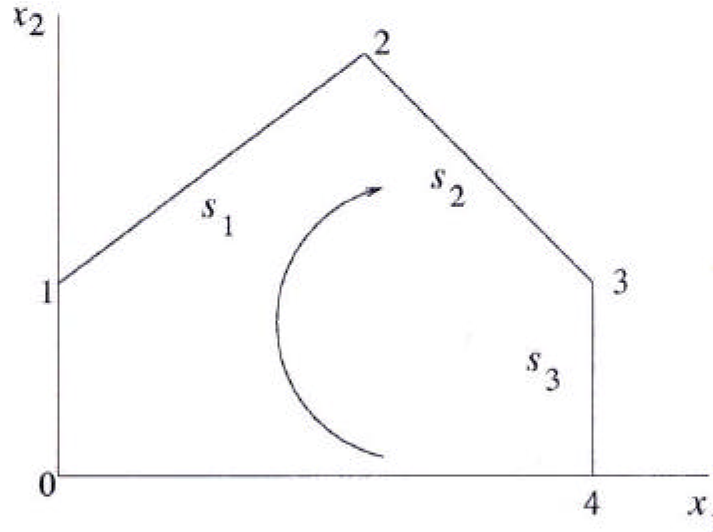
$$
\begin{aligned}
\sigma(x) &= \{j : x_j > 0 \text{ in some optimal solution}\} \\
&= \{j : d_j = 0 \text{ in every optimal solution}\} \\
&\supseteq \{j : x_j > 0 \text{ in some optimal basis (must contain } x_j)\} \\
\sigma(d) &= \{j : d_j > 0 \text{ in some optimal solution}\} \\
&= \{j : x_j = 0 \text{ in every optimal solution}\} \\
&\supseteq \{j : d_j > 0 \text{ in some optimal basis (must not contain } x_j)\} \\
\sigma(s) &= \{i : s_i > 0 \text{ in some optimal solution}\} \\
&= \{i : \pi_i = 0 \text{ in every optimal solution}\} \\
&\supseteq \{i : s_i > 0 \text{ in some optimal basis (must contain } s_i)\} \\
\sigma(\pi) &= \{i : \pi_i > 0 \text{ in some optimal solution}\} \\
&= \{i : s_i = 0 \text{ in every optimal solution}\} \\
&\supseteq \{i : \pi_i > 0 \text{ in some optimal basis (must not contain } s_i)\}
\end{aligned}
$$

Last relation in each case is equality if the optimality region is bounded.

What matters and corresponds to "verifiable" fact is structure and relationship.

— Richard Courant and Harold Robbins

Example — Vary c



c-range	$\sigma(x)$	$\sigma(d)$	$\sigma(s)$	$\sigma(\pi)$
0	$\emptyset$	$\{1,2\}$	$\{1,2,3\}$	$\emptyset$
$0-1$	$\{2\}$	$\{1\}$	$\{1,2,3\}$	$\emptyset$
1	$\{2\}$	$\{1\}$	$\{2,3\}$	$\{1\}$
$1-2$	$\{1,2\}$	$\emptyset$	$\{2,3\}$	$\{1\}$
2	$\{1,2\}$	$\emptyset$	$\{3\}$	$\{1,2\}$
$2-3$	$\{1,2\}$	$\emptyset$	$\{1,3\}$	$\{2\}$
3	$\{1,2\}$	$\emptyset$	$\{1\}$	$\{2,3\}$
$3-4$	$\{1,2\}$	$\emptyset$	$\{1,2\}$	$\{3\}$
4	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$
$4-0$	$\{1\}$	$\{2\}$	$\{1,2,3\}$	$\emptyset$

Need for the Optimal Partition

...When we need (or want) to know whether a variable is positive in *some* optimal solution.

Example 1: Job Scheduling (Critical Path Problem)

Example 2: Peer Group Identification

Example 3: Finding all Implied Equalities

Example 4: Diagnosing Infeasibility with IIS Isolations

Example 5: Assignment Problem

Example 6: Absolute Value Targets

Example 7: Multiple Objectives

There is nothing more practical than a good theory.

— Harvey M. Wagner

Example 1: Job Scheduling (Critical Path Problem)

Given: n jobs with durations to perform tasks, $\{t_j\}$; precedence relations, $P = \{<i,j>\}$, where job i must finish before job j can start.

Find: Start times of jobs to minimize total completion time, T.

LP: x_j = start time of job j; add jobs 0 and $n+1$ with $t_0 = t_{n+1} = 0$; add $<0,j>, <j,n+1>$ to P $\forall j$.

$$\min T = x_{n+1} - x_0 : x_j - x_i \geq t_i \text{ for } <i,j> \in P.$$

Dual: $\max \sum_{<i,j>\in P} \pi_{ij} t_i : \pi \geq 0,$

$$\sum_{<i,k>\in P} \pi_{ik} - \sum_{<k,j>\in P} \pi_{kj} = \begin{cases} -1 & \text{if } k = 0, \\ 1 & \text{if } k = n+1, \\ 0 & \text{if } 1 \leq k \leq n. \end{cases}$$

$\Leftrightarrow$ Longest Path Problem (each longest path = *critical path*.

Given a basic optimal solution, we can say the following:

- Critical jobs are identified by the (one) critical path.
- Reducing completion time of some critical job is necessary, but not sufficient, to reduce the total completion time.

Given an interior optimal solution, we can say the following:

- Critical jobs are identified as one that is in *some* critical path.
- Reducing completion time of some critical job is necessary, but not sufficient, to reduce the total completion time (just as in a basic optimum).
- Unlike a basic optimum, we have a sufficient condition to reduce total completion time: reduce the completion times of all critical jobs.

Information from interior solution **dominates** information from basic solution.

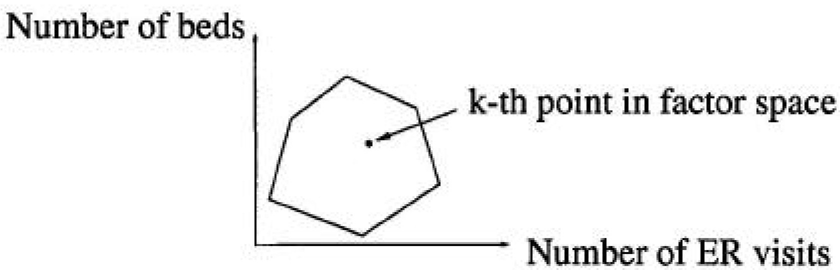
Example 2: Peer Group Identification in Data Envelope Analysis

Given: n hospitals, each with m factor values.

Find: how well a particular hospital (k-th) is doing, relative to the others, and the associated *peer group* with which the comparison is made.

LP: $\min cx : x \geq 0, \sum_j x_j = 1, \sum_j A_{ij}x_j \geq A_{ik}$ for $i \in G$,
$\sum_j A_{ij}x_j \leq A_{ik}$ for $i \in L$.

A_{ij} = value of i-th factor in hospital j;
G = performance (e.g., number of ER visits);
L = resources (e.g., number of beds).



- x determines a point in the convex hull of the factors of the hospitals.
- Factor constraints in L ensure k-th hospital has at least as many resources.
- Factor constraints in G ensure k-th hospital provides at least the same quantity and quality of health care.
- Objective is usually cost.

Reference:

R.C. Morey, D.J. Fine, S.W. Loree, D.L. Retzlaff-Roberts, and S. Tsubakitani, 1992. The Trade-off Between Hospital Cost and Quality of Care: An Exploratory Empirical Analysis, *Medical Care* 30:8, 677-698.

Peer group of the k-th hospital $= \sigma(x^*)$.

- Non-unique basic solution can give misleading evaluation.
- Unique partition better fits the meaning of a peer group.

Example 3: Finding All Implied Equalities

Given: $S = \{Ax \geq b\}$.
Find: $\{i : Ax \geq b \Rightarrow A_i x = b_i\}$.
LP: $\max \pi b : \pi A = 0, \pi \geq 0$.

Some Facts

- LP unbounded $\Rightarrow$ S has no feasible solution.
- If $\pi^* b = 0$, S is feasible and $i \in \sigma(\pi^*) \Rightarrow A_i x \geq b_i$ is an implied equality.

Suppose S has a feasible solution. Then, $\sigma(\pi^*)$ contains *all* implied equalities if the LP solution is strictly complementary.

- $\sigma(\pi^*)$ from a degenerate basic solution $\Rightarrow$ must solve more LPs.
- $\sigma(\pi^*)$ from an interior solution $\Rightarrow$ done after one LP!

Reference:

R.M. Freund, R. Roundy and M.J. Todd, 1985. *Identifying the Set of Always-Active Constraints in a System of Linear Inequalities by a Single Linear Program,* Working Paper No. 1674-85 (Rev.), Sloan School of Management, MIT, Cambridge, MA.

Example 4: Diagnosing Infeasibility with IIS Isolations

Given: $S = \{Ax \geq b\} = \emptyset$.
Find: Possible cause(s).
LP: $\max \pi b : \pi A = 0, \ 0 \leq \pi \leq w$
$(= \min wv : \ Ax + v \geq b, \ v \geq 0 \ldots$ Phase 1).

Some Facts

- $\sigma(\pi) =$ IIS iff solution is an extreme point.
- $\sim \sigma(\pi)$ can be discarded iff solution is interior.
- $|\sigma(\pi)|$ can guide strategy, especially if solution is interior.

Let $w = e$ in Chinneck's *elastic program:*

$$EP(I) : \min ev : \ Ax + v \geq b, v \geq 0, v_i = 0 \text{ for } i \in I.$$

Start with $I = \emptyset$, and set $I' = I \cup \sigma(v)$ if $EP(I)$ is feasible. Stop when $EP(I)$ becomes infeasible; then, I contains an IIS.

Total effort to obtain IIS is $O(|I| + \#EP)$. An interior solution to $EP(I)$ can result in $\#EP = 1$.

$$\text{Example: } S = \{x_1 - x_2 \geq 1, -x_1 - x_2 \geq -2, x_2 \geq 1\}.$$

S, itself, is an IIS (and the only one). $EP(\emptyset)$ has 3 basic optimal solutions:

$$x^1 = (1,1), v^1 = (1,0,0); \ x^2 = (2,1), v^2 = (0,1,0); \ x^3 = (1,0), v^3 = (0,0,1);$$

$\Rightarrow \#EP = 3$ (max possible).
Interior solution $\Rightarrow \sigma(v^*) = \{1,2,3\} \Rightarrow \#EP = 1$.

Reference:

J.W. Chinneck and E.W. Dravnieks, 1991. Locating Minimal Infeasible Constraint Sets in Linear Programs, *ORSA Journal on Computing* 3:2, 157-168.

What is an IIS?

IIS = *Irreducible Infeasible Subsystem*:

$\hookrightarrow$Dropping any one constraint causes subsystem to become feasible.

IIS provides information to analyst to diagnose the cause.

Mathematical fact: some constraint in each IIS is incorrectly stated.

Example 5: Assignment Problem

Given: n people, n tasks, and their assignment costs.
Find: min cost assignment of people to tasks
LP: $\min cx : x \geq 0, \sum_i x_{ij} = 1 \,\forall j, \sum_j x_{ij} = 1 \,\forall i.$
$x_{ij} = 1$ if person i is assigned to task j.

If want any optimal assignment $\Rightarrow$ get extreme point solution.
If want to know who should be assigned to tasks $\Rightarrow$ get optimal partition.

	Stop	Look	Listen	Go
Mary		$\frac{1}{2}$	$\frac{1}{2}$	
John	$\frac{1}{2}$			$\frac{1}{2}$
Irving	$\frac{1}{2}$		$\frac{1}{2}$	
Sarah		$\frac{1}{2}$		$\frac{1}{2}$

Mary can be optimally assigned either to Look or to Listen. Externalities can be used to decide...assuming interactive decision support is used.

Example 6: Absolute Value Sensitivity

Given: $\min \sum_j |c_j x_j - f_j| : Ax = b$ ($\text{rank}(A) = m$).

Reformulate: $\min \sum_j v_j : Ax = b$

$$\left.\begin{array}{rcr} v_j + c_j x_j & \geq & f_j \\ v_j - c_j x_j & \geq & -f_j \end{array}\right\} p_j = \max\{\pi_j^{(1)}, \pi_j^{(2)}\}$$

Question: How does f_j affect the optimal value?

Basic solution $\Rightarrow$ dual price of augmented constraint = 0 or 1.

Can have $p_j = 0$ in the basis found, yet another basis can have $p_j = 1$.

$\Rightarrow$ Basic solution need not reveal whether f_j affects the objective at all ... must pivot to find out.

Interior solution $\Rightarrow p_j = 0$ if, **and only if**, $p_j = 0$ in *every* optimal solution

$\Rightarrow$ We **know** $\partial^{\pm} z(f)/\partial f_j = 0$.

$p_j > 0 \Rightarrow \partial^{\pm} z(f)/\partial f_j = 1$.

... It does not matter what the value of p_j is!

Example 7: Multiple Objectives

Given: $\min\{c^1x, \ldots, c^Mx : Ax = b, x \geq 0\}$

Find: Pareto optimal points

Lexicographic approach:

$$\begin{aligned} z^1 &= \min\{c^1x : Ax = b, x \geq 0\} \\ z^p &= \min\{c^px : Ax = b, x \geq 0,\ c^kx = z^k \text{ for } k = 1, \ldots, p-1\} \\ &\quad \text{for } p = 2, \ldots, M \end{aligned}$$

Final Optimality region $\subseteq$ Pareto optima

Optimal partitions, $\{(B^k|N^k)\}$, satisfy

$$\begin{aligned} N^1 \subseteq N^2 \ \ldots \ \subseteq N^M = \{j : x_j &= 0 \text{ in every lexico-min solution}\} \\ B^1 \supseteq B^2 \ \ldots \ \supseteq B^M = \{j : d_j &= 0 \text{ in every lexico-min solution}\} \end{aligned}$$

...More to come.

Summary of Solution Types

Basic – generated by a simplex method.

Strictly complementary – generated by an interior point method.

- Each exists if LP has an optimal solution.
- A solution is both $\Leftrightarrow$ uniquely optimal (caveate).
- Each has information for sensitivity analysis; neither is dominate.

Conversions

Strictly complementary $\Rightarrow$ Basic ("purification")

not enough – need basis to be compatible for given direction;

too much – might not need basis to find rate and range.

Basic $\Rightarrow$ Strictly complementary (to get optimal partition)

– must visit all basic optima and interrogate nonbasics (for rays).

Basic $\Rightarrow$ Basic

– to reach compatible basis for given direction of change (to get rate)

– traverse all (compatible) bases to get range.

I never met an optimum I didn't like.

— Milton M. Gutterman

Another Example of Information in an Optimal Partition

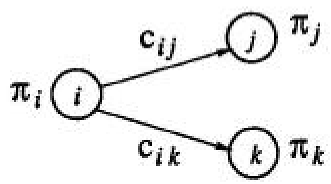
Minimum Cost Network Flow

If two *active* arcs are adjacent, the difference in prices between the two nodes always equals the difference in the arcs' transportation costs:

$$|\Delta\pi| = |\Delta c|$$

(over $\sigma(\pi) \times \sigma(x)$).

Consumer prices (common tail)

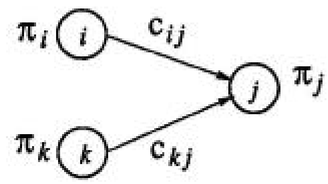


$$\pi_j - \pi_k = c_{ij} - c_{ik}$$

$$(d_{ij} = c_{ij} + \pi_i - \pi_j = 0$$

$$d_{ik} = c_{ik} + \pi_i - \pi_k = 0)$$

Producer prices (common head)



$$\pi_i - \pi_k = c_{kj} - c_{ij}$$

$$(d_{ij} = c_{ij} + \pi_i - \pi_j = 0$$

$$d_{kj} = c_{kj} + \pi_k - \pi_j = 0)$$

(*Active* could be replaced by *basic*, but the above is a **stronger statement.** If a nonbasic reduced cost = 0, there is no assurance that it is not positive in some other optimal solution.)

This constant difference is true in *every* optimal solution.

Everything should be made as simple as possible, but not simpler.

— Albert Einstein

Some Conclusions

- There exist analysis questions for which the optimal partition has more valuable information than a basic solution.
- It is costly to obtain the optimal partition using an optimizer that generates only basic solutions (so an interior method is needed).
- There are challenging frontiers in using underlying structure of an interior solution to provide useful information for sensitivity analysis (e.g., central path and animation).

The pure and simple truth is rarely pure and never simple.

— Oscar Wilde