Sensitivity Analysis

- A few concepts
  - Impulse-Response

- What do we know how to do
  - By MP class

- Related analyses
  - Consistency, Redundancy, & Implied Equalities

- Some foundations
  - Alternative/dual systems

- Some practical considerations
  - Estimating results from partial information

An 'expert' is one who doesn't know more than you but uses slides.
### Impulse-Response Queries

<table>
<thead>
<tr>
<th>Impulse</th>
<th>Data</th>
<th>Response</th>
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<tbody>
<tr>
<td>Data</td>
<td>Drive</td>
<td>Common</td>
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<tr>
<td>Solution</td>
<td>Inverse</td>
<td>Rate of substitution</td>
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*I never met an optimum I didn't like.*

– Milton M. Gutterman
## Impulse-Response Queries

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- At what rate does the objective value change when I perturb some parameter? For what range is this rate constant (or same functional form)?
- At what rate does the level (or price) change when I perturb some parameter? For what range is this rate constant (or same functional form)?

\[
\frac{d^{(\pm)}Z^*/dp}{d\rho} = k \text{ for } \rho \in [P-L, P+U]
\]

- $\rho =$ parameter
- $P =$ current value of $\rho$
- $Z^* =$ optimal objective value
- $[L, U] =$ range of change
Impulse-Response Queries

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- How can I change some parameter to cause a 10% decrease in the min cost? – e.g., decrease demand or make some inexpensive supply available
- How can I change some parameter to reach specified change in solution? – e.g., increase max oxygen to result in more glucose production.

\[ \frac{\partial (\pm Z^*)}{\partial p} = k \text{ for } p \in [P-L, P+U] \]
Impulse-Response Queries

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How can I change some parameter such that to remain in equilibrium, I must change another (specified) parameter?
– e.g., decrease demand (D) and increase some (specified) supply (S):
\[
\Delta S = k \Delta D
\]
## Impulse-Response Queries

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How does one solution value change if I force a change in some other?

\[ \partial^{(\pm)} x^*_r / \partial x^*_i \]

- Applies to phase-plane analysis.

![Graph](attachment:image.png)
Simplex Method Uses this Every Iteration

<table>
<thead>
<tr>
<th>Basic</th>
<th>Level</th>
<th>Nonbasic</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td>$x_i$</td>
</tr>
<tr>
<td>$x_r$</td>
<td>$b_r$</td>
<td>$a_{ri}$</td>
</tr>
<tr>
<td>$-Z$</td>
<td></td>
<td>$d_i$</td>
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\[ x_r = b_r - a_{ri} x_i \]

rate of substitution
Qualitative Analysis

- Given directions of change of parameter, find directions of change of solution
- Find qualitative relations among variables (degrees of separation among metabolites or reactions)
- Find stability properties (not numbers)
- Find pathways of certain interest

*Modeling is about insight, not numbers.*
– Arthur M. Geoffrion
A Quick Tour of What We Know

- Linear Programming (LP)
- Nonlinear Programming (NLP)
- Integer Programming & Combinatorial Optimization (IP/CO)
- Mixed-Integer Linear Programming (MILP)
- Mixed-Integer Nonlinear Programming (MINLP)

The pure and simple truth is rarely pure and never simple.
– Oscar Wilde
LP

- Basic solution
  - Compatibility theory
- Interior solution
  - Optimal partition
- General case
  - Character of solution

Mostly well understood, but algorithms not perfect

Qualitative analysis strongest for network models, then Leontief

MOLP: Objective space gives important insights
NLP

- Lagrange multipliers
  - Marginal analysis with convexity; “rapid” re-optimization
- Dynamic programming
  - Inherently parametric; needs separability & low dimension
- Pooling problem (bilinear constraints)
  - Exploit geometry to overcome non-convexity
  - Raised new concept – *Essential* objects (pools/reactions)

_Sometimes wrong, but never in doubt._
- Michael Evans (Economics forecaster)
IP/CO

- Generally NP-hard
  - Optimizers do not provide automatic support beyond LP

- Special focus on problem structures
  - Scheduling, TSP, covering, packing, …

- Computational logic
  - Horn clauses: if A then B (single antecedent & consequent)

- New definitions
  - Stability regions; ties with algorithm/heuristic used

- Visualization
  - Diagrammatic; Iconic; Animation
MILP

➤ Loses structural information
  - Preserve logic of IP part (binary variables to control fluxes)

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<tr>
<th>Logical</th>
<th>Algebraic</th>
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<tr>
<td>x=0 → y=0</td>
<td>x – y ≥ 0</td>
</tr>
<tr>
<td>x=0 → y=1</td>
<td>x + y ≥ 1</td>
</tr>
<tr>
<td>x=1 → y=0</td>
<td>x + y ≤ 1</td>
</tr>
<tr>
<td>x=1 → y=1</td>
<td>x – y ≤ 0</td>
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MINLP

- No theory; few special algorithms
  (I. Grossman did some things for specific problems)

*Every new body of discovery is mathematical in form, because there is no other guidance we can have.*
Other Forms

- Multiple objectives
- Goal programs
- Fuzzy programs
- Stochastic programs
- Randomized programs
- Semi-definite programs
Summary of SA Capability

- **Linear Programming (LP)**
  - Lots known; All queries; Must be careful

- **Nonlinear Programming (NLP)**
  - Only special cases (convex quadratic; bilinear)

- **Integer Programming &
  Combinatorial Optimization (IP/CO)**
  - Hard, but some good results, using logical structure

- **Mixed-Integer Linear Programming (MILP)**
  - Use IP/CO methods

- **Mixed-Integer Nonlinear Programming (MINLP)**
  - Uncharted
Consistency, Redundancy, and Implied Equalities

System: \( S = \{ Ax \geq b \} \)

Polyhedron: \( P(S) = \{ x: Ax \geq b \} \)

Subsystems: \( S(I) = \{ A_i \cdot x \geq b_i \text{ for } i \in I \} \)
\( S_i = \{ A_k \cdot x \geq b_k \text{ for } k \neq i \} \)

Inconsistent: \( P(S) = \emptyset \)

Redundant: \( P(S_i) = P(S) \)

Strongly Redundant: \( x \in P(S_i) \rightarrow A_i \cdot x > b_i \)

Implied Equality: \( A_i \cdot x = b_i \text{ for all } x \in P(S) \)

A model is to an analyst as a magnifying glass is to Sherlock Holmes – it illuminates clues.
Example

\[ S = \{0 \leq x_1, x_2 \leq 1, \ x_1 + x_2 \geq \beta\} \]

- **Redundant**
  \[ \beta = 0 \]
  \[ \beta > 0 \]

- **Implied equality**
  \[ \beta = 2 \]

- **Inconsistent**
  \[ \beta > 2 \]
Foundation = Dual system

\[ S^d = \{ y \geq 0, yA = 0, yb \geq 0 \} \]

Example

\[
\begin{align*}
    x_1 & \geq 0 \\
x_2 & \geq 0 \\
-x_1 & \geq -1 \\
-x_2 & \geq -1 \\
x_1 + x_2 & \geq \beta \\
y_1, y_2, y_3, y_4, y_5 & \geq 0 \\
y_1 - y_3 + y_5 & = 0 \\
y_2 - y_4 + y_5 & = 0 \\
-y_3 - y_4 + \beta y_5 & \geq 0
\end{align*}
\]

A study of economics usually reveals that the best time to buy anything is last year.

– Marty Allen
# Certification

Property of $S$ is true $\iff S^*$ is consistent

<table>
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<tr>
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<th>$S^*$</th>
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<td>Redundant</td>
<td>$S^d {y_i \geq 0} &amp; {y_i &lt; 0}$</td>
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<td>$S^d {y_i \geq 0} &amp; {y_i &lt; 0, \ y_b &gt; 0}$</td>
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\[
\begin{align*}
y_1, \ y_2, \ y_3, \ y_4, \ y_5 & \geq 0 \\
y_1 - y_3 + y_5 & = 0 \\
y_2 - y_4 + y_5 & = 0 \\
- y_3 - y_4 + \beta y_5 & \geq 0
\end{align*}
\]
Certification of Redundancy

Property of S is true ⇔ S* is consistent

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$\beta = 0$

choose $y = (1, 1, 0, 0, -1)$

\[
y_1, y_2, y_3, y_4, y_5 \geq 0 \quad y_5 < 0
\]

\[
y_1 - y_3 + y_5 = 0
\]

\[
y_2 - y_4 + y_5 = 0
\]

\[
-y_3 - y_4 \cdot \geq 0 \quad \rightarrow y_3 = y_4 = 0
\]
Certification of Strong Redundancy

Property of $S$ is true $\leftrightarrow S^*$ is consistent

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$\beta < 0$

choose $y = (1, 1, 0, 0, -1)$

certifies strong redundancy because $y_b = -\beta > 0$

$y_1, y_2, y_3, y_4, y_5 \geq 0 \quad y_5 < 0$

$y_1 - y_3 + y_5 = 0$

$y_2 - y_4 + y_5 = 0$

$-y_3 - y_4 + \beta y_5 \geq 0$
## Certification of Implied Equality

Property of $S$ is true $\leftrightarrow S^*$ is consistent

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\[
\begin{align*}
 y_1, y_2, y_3, y_4, y_5 & \geq 0 \\
y_1 - y_3 + y_5 & = 0 \\
y_2 - y_4 + y_5 & = 0 \\
- y_3 - y_4 + \beta y_5 & \geq 0
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Certification of Implied Equality

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$\beta = 2$

choose $y = (0, 0, 1, 1, 1)$ plays no role in implied equality

$y_1, y_2, y_3, y_4, y_5 \geq 0$

$y_1 - y_3 + y_5 = 0$

$y_2 - y_4 + y_5 = 0$

$- y_3 - y_4 + 2y_5 \geq 0$
Certification of Inconsistency

Property of $S$ is true $\iff S^*$ is consistent

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$\beta > 2$
choose $y = (0, 0, 1, 1, 1)$

plays no role in inconsistency

$y_1, y_2, y_3, y_4, y_5 \geq 0$
$y_1 - y_3 + y_5 = 0$
$y_2 - y_4 + y_5 = 0$
$- y_3 - y_4 + \beta y_5 \geq 0$


**Certification of Inconsistency**

Property of S is true $\iff$ S* is consistent

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$\beta > 2$

choose $y = (0, 0, 1, 1, 1)$

certifies inconsistency because $y_b = \beta - 2 > 0$

\begin{align*}
\beta > 2 & \\
\text{choose } y = (0, 0, 1, 1, 1) & \\
\text{certifies inconsistency because } y_b = \beta - 2 > 0 & \\

y_1, y_2, y_3, y_4, y_5 \geq 0 & \\
y_1 - y_3 + y_5 = 0 & \\
y_2 - y_4 + y_5 = 0 & \\
- y_3 - y_4 + \beta y_5 \geq 0 & \\
\end{align*}
Certificates Obtained by LP

\[
\max yb: \ y \in P(S^\sim), \ \Sigma_i y_i = 1
\]

\[\Rightarrow\] every \(i\) for which \(y^*_i > 0\) is an implied equality

Normalization to have \(y \neq 0\)

Interior Solutions Certify All at Once with \(\sigma(y)\)

\(y^* \in \arg\max\{y^T b: \ yA = 0, \ y \geq 0, \ \Sigma_i y_i = 1\}\)

\[\Rightarrow\] every \(i\) for which \(y^*_i > 0\) is an implied equality

\(y^*\) interior \(\Rightarrow \sigma(y) = \) set of \textit{all} implied equalities of \(Ax \geq b\)